University College London
DEPARTMENT OF MATHEMATICS
Mid-Sessional Examinations 2010
Mathematics 1101
Friday 15 January 2010 2.30 – 4.30 or 4.00 – 6.00

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination*

1. (a) State what it means for a real sequence to converge.

   (b) State and prove the Sandwich theorem for sequences.

   (c) Prove that $2^n > n$ for $n \in \mathbb{N}$.

   (d) Use the definition of convergence (not the combination theorem or the sandwich theorem) to show that
$$\lim_{n\to\infty} \frac{3 \cdot 2^n + 1}{2^n + 5} = 3.$$

2. (a) State the definition of $\lim_{x\to a^+} f(x) = l$.

   (b) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by
$$f(x) = \begin{cases} (x-1)^2 + 5, & (x < 1), \\ 1, & (x = 1), \\ 3x + 2, & (x > 1). \end{cases}$$

   (i) Using only the definitions, i.e., using ϵ and δ, show that
$$\lim_{x\to 1-} f(x) = 5, \quad \lim_{x\to 1+} f(x) = 5.$$

   (ii) Is the function continuous at $x = 1$?

   (c) State the Bolzano–Weierstrass Theorem.

   (d) Let f be continuous on the compact interval $[a, b]$. Prove that f is bounded on $[a, b]$.

3. (a) State the Least Upper Bound Principle (continuum property).

   (b) If they exist, find with justification the sup, inf, max and min of the set
$$S = \{2^{-k} + 1 : k \in \mathbb{N}, k \geq 2\}.$$

   (c) Define what it means for a sequence to be Cauchy.

   (d) State the General Principle of Convergence.

   (e) Prove that every Cauchy sequence is bounded.

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4. (a) State and prove the Cauchy–Schwarz inequality.

(b) State and prove the Intermediate Value Theorem.

(c) Let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be continuous functions with

$$f(a) > g(a), \quad f(b) < g(b).$$

Prove that we can find a $\xi \in (a, b)$ with $f(\xi) = g(\xi)$.

(d) Let $f(x)$ be a continuous function on $[0, \infty)$ with $f(0) = 0$. We assume we can find a constant k such that

$$\ln|16 - f(x)| = -x + k, \quad \forall x \in [0, \infty).$$

Show that

$$f(x) = 16 - 16e^{-x}, \quad \forall x \in [0, \infty).$$

5. (a) Prove that the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges.

(b) State the alternating series test.

(c) Determine with explanations whether the following series converge or diverge.

$$\sum_{n=1}^{\infty} \frac{3n+1}{2n^3 - n}, \quad \sum_{n=1}^{\infty} \frac{(3n)!}{(n)!(2n)!}, \quad \sum_{n=1}^{\infty} 2^n \left(\frac{n}{n+1}\right)^{n^2}.$$

(d) If the series $\sum_{n=1}^{\infty} a_n^2$ converges, show that the series

$$\sum_{n=1}^{\infty} \frac{a_n}{n^{3/2}}$$

converges absolutely.

6. (a) State the Arithmetic Mean – Geometric Mean Inequality for n non-negative numbers $a_1, a_2, \ldots, a_n$.

(b) Prove that the sequence $x_n = \left(1 + \frac{1}{n}\right)^n$ is increasing.

(c) Prove that the sequence $y_n = \left(1 + \frac{1}{n}\right)^{n+1}$ is decreasing.

(d) Show that $x_n < y_n$. Deduce that x_n is bounded above, while y_n is bounded below. Conclude that they have the same limit.

END OF PAPER